**Q1D Evolution equations for t, r, tʹ, rʹ**

So we can get the evolution equations for t, r, tʹ, rʹ in 1D. Tartakovsky shows us how to do the same thing for Q1D. I have put more exploratory calculations in the GDMPK II research folder.

**Getting t and rʹ evolution equations, as function of z2**

Before we begin, let’s examine the change in phase convention he uses. In his notation, the two wavefunctions are ψ´L(z-z­1) and ψ´R(z-z­2). We can massage these expressions into our usual one like this:



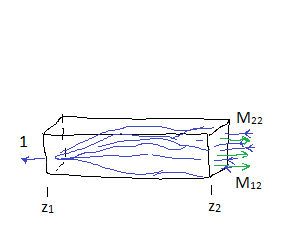
So new M´ is:



And so,



where M´ is the M we’ll be dealing with in his approach, and M is our usual M. Once we get one we have the other. Gonna not use primes anymore, and everything will refer to his ψ and his M. So we’re going to consider the following scenario: a wavefunction traveling right to left through a medium between z1 and z2 in such fashion that it transmits (with unit flux) through only one channel, n, on the left. We will call this wavefunction ψ(n)(x,y,z).



We shall consider the situation from left to right nonetheless. And we’d like to write out the Schrodinger equation from left to right in terms of the transverse eigenbasis Φm(x,y). Note that even inside the disordered region, the wavefunction will lie within the same transverse basis as outside – otherwise there would be no way to stitch together the wavefunctions outside and inside the disordered region. So first let’s consider the wavefunction on the left of the disordered region. It will be:



We’ll note that the presence of the odd phase factor – but it’s necessary for our ultimate aim, which is to develop an equation for M itself. In the transverse basis would be:



On the other side of the disordered region it would look like? Well we have the transfer matrix which connects states from right to left.



We want a+ = 0, whereas a- will correspond to just a single channel (n). So:



And so we can write the wavefunction as:



(somehow the phase factors changed? – yes because new M´) and in the transverse basis we have:



And now we can look inbetween the leads. We’ll start with Schrodinger’s equation,



and to perform averages over the random potential later, we will assume that it follows:



i.e. the Born approximation. This would mean that we will only consider scattering off of an impurity twice? Now to connect this to a 1D problem, let’s put the Schrodinger equation in the transverse basis. So we can say:



where we’ve integrated by parts to get kT. And next we’ll insert a resolution of identity (transverse) between U and ψ.



And so we have:



(implicit summation on the α). We could more concisely put this in matrix form:



And even though M and eik(z-z1)/√k|| were not being multiplied as matrices per se´, we can still write it as a matrix multiplication since we get the same result. So these two highlighted equations constitute an ODE and boundary conditions for the wavefunction . On the other hand, we can think of this as an initial value problem, describing how the wavefunction evolves from ~ 1mn at z = z1 into ~ M12 and ~ M22 by z = z2. Now we want to develop an ODE for the transfer matrix elements M11 and M12. Or really, M12 and M22. To that end, let’s construct a wavefunction for which those elements will be inherent (and transparent).

Let’s see if we can write an evolution equation that will decouple the M12 and M22 parts. We want equations involving two quantities that evolve from something to M11 and from something else to M12 respectively. So, the two parts are:



We should be able to write out two equations for each of these variables. For instance, consider equation y´´ = 4. If we make definition: u = y, v = y´, we can write our ODE as: v´ = 4; u´ = v. So let’s do the same with the Schrodinger equation and these u’s. I’ll start by taking derivative of each:



Then we will solve for ψ and ∂ψ/∂z from the definitions of u± and plug them into the two equations above.



Inserting these into our expression for ∂u/∂z we get:



and this splits into the pair of equations:



and we’ll observe that these obey boundary conditions:



whereas at the other point,



Doing the same with the other we will get:



If you have some discomfort evaluating the boundary conditions at z1 and z2 exactly, and not using the asymptotic form, remember that this is actually how you did it in all the other scattering problems. You don’t need the actually asymptotic form of the wavefunction per se´. We can also now see why he defines his wavefunction with the exponential (z-z1), (z-z2) factors – otherwise we’d be left with residual exponential factors in the boundary conditions at the end point z2 of the form eikL, and while this wouldn’t be a problem, as we could absorb it into M, or something, it doesn’t look as nice. Next, we’ll observe that what we have is a quantity, u+ which evolves from 1 to M22 and a quantity u+ which evolves from 0 to M12 as z goes from z1 to z2. So effectively, we could say that u+ = M12 and u­- = M22, and thus we have found the evolution equations of these matrix elements. So we have:



One penultimate note. Since z2 is a variable, we may think of u+(z) and u­-(z) as M22(z) and M12(z) respectively. And so we see that when we solve these stochastic differential equations we would get an expression like u+(z) = f(L), where f would be some matrix function of L, and u-(z) = g(L). Note that we can expand these functions in a power series in L, and apparently, judging from the initial conditions, we get: f(L) = 1 + …, g(L) = 0 + … Let’s compare this to the polar representation of the transfer matrix:



We can conceive just as well that u, υ, and λ also can be written as coupled stochastic differential equations with their own evolution. And we can say that since M11 → 1, then uυ → 1 as well, and since M22 → 0, we can say that λ → 0 as well. So u, υ, λ could be written as some f(L), g(L), h(L), with the requisite N2, N2, N d.o.f. but with f(0)g(0) = 1, and h(0) = 0, etc. Now let’s get the analogous equations for the reflection and transmission matrices. So we will go back to:



and we have:



We will take the derivative of t´ and see if we can get an equation for it. But we must be careful about commuting issues. So let’s do an implicit derivative. But note we seem to be assuming a Stratonovich derivative here.



Yep. That’s why you have to be careful. So filling in the equation of motion for M22 we get:



And now using t´ = tT, and r´T = r´, we can say:



Now to complete the equations we need an evolution equation for r´. Again we seem to be assuming a Stratonovich derivative. We have:



It’s interesting that the r´ equation is completely uncoupled, as it was in the 1D case too. Well now grouping these together we get:



He notes that the r´ equation is entirely self-sufficient. Still it isn’t useful for analyzing the conductance apparently since there is no small quantity present in the equation to facilitate analysis. For instance, he says that as t → 0, r will go to 1 in a unitary fashion (since r†r + t†t = 1 → r+r → 1 as t → 0). Again, I think these equations are evaluated in the Stratonovich approach.

**Now t and r as a function of z1**

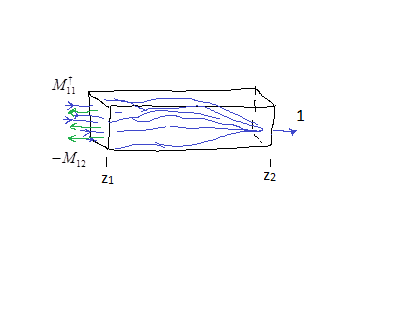
Now we want to consider a setup where the transfer matrix elements are on the left side, I think. Taking advantage of the following identities,



Then, we’ll have:



Which can be depicted like this:



So in the transverse basis, we have as boundary conditions:



In between, we of course have, as before:



We could more concisely put this in matrix form:



Let’s see if we can write an evolution equation that will decouple the M11 and M12 parts. Could try the usual:



Differentiating:



Then we will solve for ψ and ∂ψ/∂z from the definitions of u± and plug them into the two equations above.



Inserting these into our expression for ∂u/∂z we get:



and this splits into the pair of equations:



and we’ll observe that these obey boundary conditions:



whereas at the other point,



Doing the same with the other we get:



whereas at the other point,



So we have:



Filling in the M’s,



And daggering,



What about t and rʹ evolution equations?



And now using t´ = tT, we can say:



Which matches his result! Moving on to rʹ…



A few manipulations shows that [ ] is none other than t. So we have:



**Statistical Averaging**

Now we calculate a statistical average which will be useful shortly. To start with, we have:



So his φ0 = π. Note that under the assumption of weak disorder, ℓ >> 1/kF, this disorder term would be small (and so a perturbation expansion would make sense). Anyway we have:



The integral in the bracket is:



The second expression is zero. This is because iana + ibnb + icnc + idnd comes in equal and opposite pairs. If one set of signs adds up to an odd integer, then flipping all the signs will also add up to the (negative) odd integer. That will flip the sign of the result of the integral. And so all terms will cancel in equal and opposite pairs. It must cancel out anyway, b/c if it didn’t, we’d get an imaginary component to our answer. So that will leave us with only:



Filling this into our general result…



So we have:



As we determined in previous files, Δ has very weak index dependence and oscillates around 3/8 or so. And so now with the statistical averaging specified we could in principle write down an evolution equation for the probability distribution of r΄mn(z) and tmn(z), as has been done for example in the Stochastic file. In fact, the evolution equation for r΄mn(z) is decoupled from tmn(z) and so could be solved in principle, and then plugged into tmn(z) to get a decoupled equation for it. Interestingly, these equations are of finite order (2nd order as usual), not infinite, as is what happens in Mello’s stuff (well we’ve already taken the continuum limit here, so that’s to be expected). But he says that the equations are not tractable, b/c they provide no ready starting point for analysis via PT. On the other hand, if we assume that we have weak disorder, then the U terms become smaller than the others and so a perturbative expansion becomes tenable.